Conformal maps; distortion theory.

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Def $f:\Omega\to\hat{\mathbb{C}}$ - conformal, if 1) f is analytic
2) $\forall z_1,z_2\in\Omega,\ z_1\neq z_2 \Rightarrow f(z_1)\neq f(z_2)$.

f - conformal map of Ω onto $f(\Omega)$.

Thm (Riemann) Ω - simply connected, $\Omega\subset\hat{\mathbb{C}}$, $w_0\in\Omega$.
Then $\exists!$ $f:\mathbb{D}\to\Omega$ - conformal, $f(0)=w_0$, $f'(w_0)>0$. $\mathbb{D}$ - unit disk, $\mathbb{D}^*$ - $\{|z|>1\}$.

corollary. Ω_1,Ω_2 - s.c., $w_1\in\Omega_1$, $w_2\in\Omega_2 \Rightarrow \exists!\ f:\Omega_1\to\Omega_2$, $f'(w_1)>0$, $f(w_1)=w_2$.

We will study the behaviour of conformal maps inside Ω first.

Schwarz lemma: $f:\mathbb{D}\to\mathbb{D}$, $f(0)=0 \Rightarrow$ 1) $|f(z)|\le|z|$ 2) $|f'(z)|\le 1$. Equality $\Leftrightarrow f(z)=e^{i\theta}z$.

Pf. Apply maximum principle to $\varphi(z)=\frac{f(z)}{z}$ ∎

A corollary: self-maps of $\mathbb{D}$. $\mathrm{Aut}(\mathbb{D}):=\{\varphi:\mathbb{D}\to\mathbb{D}$ - conformal$\}$,

$\mathrm{Aut}(\mathbb{D})=\{e^{i\theta}\frac{z-z_0}{1-\bar z_0 z},\ z_0\in\mathbb{D},\ e^{i\theta}\in S^1\}$.

Pf. $e^{it}\frac{z-z_0}{1-\bar z_0 z}\in\mathrm{Aut}(\mathbb{D})$,

Let now $f\in\mathrm{Aut}(\mathbb{D})$, $z_0:=f(0)$. Then $h(z):=\frac{f(z)-z_0}{1-\bar z_0 f(z)}$ maps $\mathbb{D}\to\mathbb{D}$, $0\to 0$.

So both h and h^{-1} satisfy Schwarz lemma. Thus $|h'(0)|=1$, so $h(z)=e^{i\theta}z$.

So $\frac{f(z)-z_0}{1-\bar z_0 f(z)}=e^{i\theta}z$, i.e. $f(z)=e^{i\theta}\frac{z-(-e^{-i\theta}z_0)}{1-\overline{(-e^{-i\theta}z_0)}\,z}$ ∎

Hyperbolic metric: $\frac{|dz|}{1-|z|^2}$.

$\rho(z_1,z_2)=\inf_{\gamma \text{ joining } z_1,z_2}\int_\gamma\frac{|dz|}{1-|z|^2}=\log\left(\frac{1+\left|\frac{z_2-z_1}{1-\bar z_1 z_2}\right|}{1-\left|\frac{z_2-z_1}{1-\bar z_1 z_2}\right|}\right)$ Convenient to look at $\tanh\rho(z_1,z_2)=\left|\frac{z_2-z_1}{1-\bar z_1 z_2}\right|$ - pseudohyperbolic distance.

Easy to check: invariant under $\mathrm{Aut}(\mathbb{D})$:

$\rho(\varphi(z_1),\varphi(z_2))=\rho(z_1,z_2)$.

Thus, if Ω - s.c., then $\rho_\Omega(z_1,z_2):=\rho(f^{-1}(z_1),f^{-1}(z_2))$ is well defined ($f:\mathbb{D}\to\Omega$ - conformal)

(if $g:\mathbb{D}\to\Omega$ - conformal, then $g\circ f^{-1}\in\mathrm{Aut}(\mathbb{D})$, so $\rho(f^{-1}(z_1),f^{-1}(z_2))=\rho(g^{-1}(z_1),g^{-1}(z_2))$).

Hyperbolic metric in $\mathbb{H}$: $\frac{|dz|}{y}$.

Schwarz Lemma (invariant form). $f:\Omega_1\to\Omega_2$ - analytic, then
1) $\forall z_1,z_2$: $\rho_{\Omega_2}(f(z_1),f(z_2))\le\rho_{\Omega_1}(z_1,z_2)$
2) Equality for some $z_1,z_2 \Leftrightarrow f:\Omega_1\to\Omega_2$ - conformal.

Pf. Just map everything back by Riemann maps. ∎

Corollary. $f:\mathbb{D}\to\Omega$ - conformal, then $\mathrm{dist}(f(z),\partial\Omega)\le|f'(z)|(1-|z|^2)$

Pf. Let $g(w):=f^{-1}(f(z)+w\,\mathrm{dist}(z,\partial\Omega))$ well-defined, maps $\mathbb{D}\to\mathbb{D}$, $0\to z$.

$h(w):=\frac{g(w)-z}{1-\bar z g(w)}$. $h:\mathbb{D}\to\mathbb{D}$, $h(0)=0$. So $|h'(0)|\le 1$. But

$h'(0)=\frac{\mathrm{dist}(f(z),\partial\Omega)}{f'(z)(1-|z|^2)}$ ∎

Thm (Hurwitz). f_n: conformal in Ω, $f_n\to f$ (unif on compacts) $\Rightarrow$ either $f\equiv$ const in Ω or f is conformal in Ω.

Pf. 1) f is analytic.

2) Let $f(z_1)=f(z_2)$. $g_n(z):=f_n(z)-f_n(z_2)$, $g(z):=\lim_{n\to\infty}g_n(z)$. If $g\not\equiv 0$, then $\exists\gamma$ - surrounding z_1 but not z_2 such that $g(z)\neq 0$ on γ. Let $\delta:=\min|g(z)|$. $\exists N: n>N: |g_n-g|<\delta$ on γ. g has zero inside $\gamma(z_1)$, so is g_n, by Rouche Theorem. Thus $g_n(w)=g_n(z_2)$ for some w inside γ - contradicts 1-1. ∎

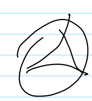
Def. Normal family of functions is precompact in $A(\mathbb{D})$ - analytic with unif. on compact. (i.e. $\forall(f_n)\ \exists(f_{n_k})$, $f_{n_k}\to f\in A(\mathbb{D})$)

Easy Thm. Any locally bounded family is precompact in $A(\mathbb{D})$.

Pf. Arzela-Ascoli + Cauchy ∎

Thm (Montel). Let $\mathcal{F}\subset A(\mathbb{D})$ omits two points, i.e. $\exists a\neq b\in\mathbb{C}: \forall f\in\mathcal{F}\ f^{-1}(a)=\emptyset$, $f^{-1}(b)=\emptyset$. Then f is normal.

Pf. $\exists$ covering $\pi:\mathbb{D}\to\mathbb{C}\setminus\{0,1\}$ - modular function.



Can define hyperbolic distance.

Let f_n: $f_n(0)$ converges (can always select converging subsequence in $\hat{\mathbb{C}}$).

$f_n(0) \to z \in \hat{\mathbb{C}}$. If $z \in \{0, 1, \infty\}$, by Schwarz lemma, $f_n(w) \to z$ $\forall w$.
Otherwise, lift f_n to the cover; $g_n : (\mathbb{D}, 0) \to (\mathbb{D}, \xi_n)$. Bounded, so $\exists g_{n_k}$ - converging,

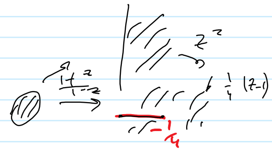
$$\begin{array}{ccc} & & \mathbb{D} \\ & \overset{g_n}{\nearrow} & \downarrow \pi \\ \mathbb{D} & \xrightarrow{f_n} & \mathbb{C}\setminus\{0,1,\infty\} \end{array}$$

so is $f_{n_k} = \pi \circ g_{n_k}$ ∎

Normalization:

$(S) = \{f$ - conformal in $\mathbb{D}$, $f(0) = 0$, $f'(0) = 1\} = \{f(z) = z + \sum_{n \geq 1} a_n z^n\}$.
$(\Sigma) = \{f$ - conf in $\mathbb{D}^*$, $f(z) = z + b_0 + \frac{b_{-1}}{z} + \ldots$ near $\infty\}$.
Connection: $f(z) \in (S) \iff \frac{1}{f(\frac{1}{z})} \in (\Sigma)$
$g(z) \in (\Sigma)$, $c \notin g(\mathbb{D}^*) \iff \frac{1}{g(1/z) - c} \in (S)$

Examples (important): 1) Köbe function $k(z) = \frac{z}{(1-z)^2} = z + 2z^2 + 3z^3 + \ldots$



(since $k(z) = \frac{1}{4}\left[\left(\frac{1+z}{1-z}\right)^2 - 1\right]$

Remark. $k_\alpha(z) := \frac{1}{2\alpha}\left[\left(\frac{1+z}{1-z}\right)^{\alpha} - 1\right] \in S$ for $\alpha \in (0, 2]$.

2) Joukowsky transform:
$f(z) = z + \frac{1}{z} \in (\Sigma)$, $f : \mathbb{D}_* \to \mathbb{C} \setminus [-2, 2]$.

Transforms preserving (S)

1) nth root transform: $f \to z\sqrt[n]{\frac{f(z^n)}{z^n}}$ - n fold symmetrization. $g(z_1) = g(z_2) \Rightarrow g^n(z_1) = g^n(z_2) \Rightarrow z_1^n = z_2^n$. But $g(z_1) = z_1\sqrt{\ldots}$ $z_2\sqrt{\ldots} = g(z_2)$.
2) Köbe transform: $\tau \in \mathrm{Aut}(\mathbb{D})$, $f_\tau := \frac{f\circ\tau - f\circ\tau(0)}{(f\circ\tau)'(0)} \in (S)$.
In particular, for $\tau(z) = e^{i\theta}z$, $f_\tau = e^{-i\theta} f(e^{i\theta}z)$
3) Symmetry: $f \to \overline{f(\bar{z})}$.

Thm (Köbe) (S) is compact in $A(\mathbb{D})$.
Pf. Closed by Hurwitz (since $f_n \to f \Rightarrow f'_n(0) \to f'(0)$).
Normality: By Hurwitz: $\forall f \in (S)$ $\exists$ $a(f), b(f) \notin f(\mathbb{D})$, $|a(f)| = 1$, $|b(f)| = 2$.
Take (f_n). $A_n := \frac{(z - a_n)}{b_n - a_n}$ $A_n \circ f_n : \mathbb{D} \to \mathbb{C} \setminus \{0, 1, \infty\}$. So, by Montel,
$\exists$ conv. subsequence $A_{n_k} \circ f_{n_k}$. By passing to further subsequence, $a_{n_k} \to a$, $b_{n_k} \to b$, $a \neq b$,
since $|a| = 1$, $|b| = 2$. So $A_{n_k} \to A$, so $A_{n_k} \circ f_{n_k} \to A \circ f \Rightarrow f_{n_k} \to f$ ∎

Harder Theory: Köbe - Biberbach.

Step 1: (Gronwall Area Thm).
$F \in (\Sigma) \Rightarrow \sum_{n \geq 1} n|b_{-n}|^2 \leq 1$.
Pf. Let $R > 1$, $\Omega_R := \mathbb{C} \setminus F(R\mathbb{D})$.
$\gamma_R := F(RS^1)$ (well-defined for $R > 1$).
By Green's formula, $\mathrm{Area}(\Omega_R) = -\frac{i}{2}\int_{\gamma_R} w\,d\bar{w} = -\frac{i}{2}\int_0^{2\pi} F(Re^{i\theta})\overline{\frac{\partial F}{\partial\theta}(Re^{i\theta})}\,d\theta$.
$F(Re^{i\theta}) = Re^{i\theta} + \sum b_{-n}R^{-n}e^{-in\theta}$
$\frac{\partial F}{\partial\theta} = i\left(Re^{i\theta} - \sum n b_{-n} R^{-n} e^{-in\theta}\right)$.
Since $\int f\bar{g} = \pi\sum a_n b_n$ - Parceval, $\mathrm{Area}(\Omega_R) = \pi\left(R^2 - \sum n\frac{|b_n|^2}{R^{2n}}\right)$. Let $R \to 1$ ∎

Remark. Equality $\iff A(R) \to 0 \iff \mathrm{Area}(\Omega_1) = 0$.

Corollary. $|b_{-1}| \leq 1$, $= 1 \iff F$ is a shift-rotation of Joukovski function ∎

Thm. (Bieberbach) $|a_2| \leq 2$ for $f \in S$

Remark. de Branges, 1984, finally proved that $|a_n| \leq n$, so Köbe is extremal.

Pf. Let $g(z) := z\sqrt{\frac{f(z^2)}{z^2}} = 1 + \frac{a_2}{2}z^2 + \ldots$
$G(z) = \frac{1}{g(\frac{1}{z})} = z + \frac{a_2}{2}\cdot\frac{1}{z} + \ldots$ so $|a_2| \leq 2$ ∎

Important Köbe Thm. $f \in (S) \Rightarrow \frac{1}{4}\mathbb{D} \subset f(\mathbb{D})$.
Pf. $w \notin f(\mathbb{D}) \Rightarrow g(z) := \frac{wf(z)}{w - f(z)} = z + (a_2 + \frac{1}{w})z^2 + \ldots \in (S)$
By Bieberbach, $|a_2 + \frac{1}{w}| \leq 2$, $|a_2| \leq 2 \Rightarrow |\frac{1}{w}| \leq 4 \Rightarrow |w| \geq \frac{1}{4}$ ∎

Remark. Exact: Köbe function.

Corollary 1 $f : \mathbb{D} \to \Omega$ - conformal $\Rightarrow \frac{1}{4}|f'(z)|(1 - |z|^2) \leq \mathrm{dist}(f(z), \partial\Omega) \leq |f'(z)|(1 - |z|^2)$.
Pf. RHS - already derived from Shwarz.
LHS: for $z_0 \in \mathbb{D}$, $g(z) := f\left(\frac{z + z_0}{1 + \bar{z}_0 z}\right) - f(z_0) \in (S)$, so $w \notin f(\mathbb{D}) \Rightarrow \frac{|w - f(z_0)|}{|f'(z_0)|(1 - |z_0|^2)} \geq \frac{1}{4}$ ∎

Cor 1. $f: \mathbb{D} \to \Omega$ – conformal $\Rightarrow \frac{1}{4}|f'(z)|(1-|z|^2) \le \operatorname{dist}(f(z), \partial\Omega) \le |f'(z)|(1-|z|^2)$.

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LHS: for $z_0 \in \mathbb{D}$, $g(z) := \dfrac{f\left(\frac{z+z_0}{1+\bar{z}_0 z}\right) - f(z_0)}{f'(z_0)(1-|z_0|^2)} \in (S)$, so $w \notin f(\mathbb{D}) \Rightarrow \dfrac{|w-f(z_0)|}{|f'(z_0)|(1-|z_0|^2)} \ge \frac{1}{4}$ ∎

Corollary 2 $\varphi: \Omega_1 \to \Omega_2$ – conformal $\Rightarrow \frac{1}{4} \le \dfrac{\operatorname{dist}(\varphi(z), \partial\Omega_2)}{|\varphi'(z)| \operatorname{dist}(z, \partial\Omega_1)} \le 4$ ∎

Def. Quasi-hyperbolic metric:

$$Q_\Omega(z_1, z_2) = \inf_{\gamma_{z_1 z_2}} \int \frac{|dw|}{\operatorname{dist}(w, \partial\Omega)}.$$

Cor. 3 $\rho_\Omega(z_1, z_2) \le Q_\Omega(z_1, z_2) \le 4 \rho_\Omega(z_1, z_2)$.

Whitney squares are almost conf. invariant.

Thm (Bieberbach inequality). $f \in (S)$, $|z| = r$. Then

$$\left| \frac{z f''(z)}{f'(z)} - \frac{2r^2}{1-r^2} \right| \le \frac{4r}{1-r^2} \qquad \text{(Exact: köbe function).}$$

Pf. $\tau(\zeta) := \dfrac{\zeta + z}{1 + \bar{z}\zeta}$ $\quad g(\zeta) := f_\tau(\zeta) = \zeta + a_2\zeta^2 + \dots$, $a_2 = \frac{1}{2}\left[\underbrace{(1 - z\bar{z})}_{(1-|z|^2)} \frac{f''(z)}{f'(z)} - 2\bar{z}\right]$

$|a_2| \le 2$. Multiply by $\dfrac{2z}{1-|z|^2}$ ∎

Observe $\operatorname{Re} \dfrac{z g'(z)}{|z|} = \dfrac{\partial \operatorname{Re} g}{\partial r}$ $\quad \operatorname{Im} \dfrac{z g'(z)}{|z|} = \dfrac{\partial \operatorname{Im} g}{\partial r}$.

$\dfrac{f''}{f'} = (\log f')'$; $\quad \dfrac{\partial \log|f'|}{\partial r} = \dfrac{\partial \operatorname{Re} \log f'}{\partial r} = \operatorname{Re} \dfrac{z (\log f')'}{|z|}$

so we get, $\dfrac{2r-4}{1-r^2} \le \dfrac{\partial}{\partial r} \log|f'(z)| \le \dfrac{2r+4}{1-r^2}$. Integrate $\int_0^r$ to get

Thm (Distortion Thm) $\dfrac{1-r}{(1+r)^3} \le |f'(z)| \le \dfrac{1+r}{(1-r)^3}$.

If we use Im part, we get

$-\dfrac{4}{1-r^2} \le \dfrac{\partial}{\partial r} \arg f'(re^{i\theta}) \le \dfrac{4}{1-r^2} \Rightarrow$ Thm (Rotation Thm) $|\arg f'(z)| \le 2 \log \dfrac{1+r}{1-r}$.

Imprecise! Precise: $|\arg f'(re^{i\theta})| \le \begin{cases} 4 \arcsin r, & r < \frac{1}{\sqrt{2}} \\ \pi + \log\left(\frac{r^2}{1-r^2}\right), & r \ge \frac{1}{\sqrt{2}} \end{cases}$ – using Löwner chains

Moreover, can get

Thm (Growth Thm) $\dfrac{r}{(1+r)^2} \le |f(z)| \le \dfrac{r}{(1-r)^2}$

Pf.

Integrate again: $|f(z)| \le \int |f'(se^{i\theta})| \, ds \le \int_0^r \dfrac{1+r}{(1-r)^3} = \dfrac{r}{(1-r)^2}$.

Lower bound: let z_0 be the point with $\{|z| = r\}$ with the smallest $|f(z_0)|$.

Then the segment $[0, f(z_0)] \subset f(r\mathbb{D})$, so $\gamma := f^{-1}([0, f(z_0)]) \subset r\mathbb{D}$; joins 0 to z_0.

$\therefore |f(z_0)| = \int_{f(\gamma)} |dw| = \int_\gamma |f'(z)| \, |dz| \ge \int_\gamma \dfrac{1-|z|}{(1+|z|)^3} |dz| \ge \int_\gamma \dfrac{1-|z|}{(1+|z|)^3} \, d|z| = \dfrac{r}{(1+r)^2}$ ∎

Thm $\dfrac{1-r}{r(1+r)} \le \dfrac{|f'(z)|}{|f(z)|} \le \dfrac{1+r}{r(1-r)}$

Pf. Use $\tau(\zeta) = \dfrac{\zeta + z}{1 + \bar{\zeta} z}$, & $f_\tau(\zeta)$, use at $\zeta = -z_0$, to see that

$$\left| z \frac{f'(z)}{f(z)} \right| = \frac{r}{(1-r^2)|f_\tau(-z)|}$$ and use growth Thm ∎

Bloch norm

Def $\|b\|_{\mathcal{B}} = \sup (1-|z|^2)|b'(z)|$ ~ Bloch norm. (actually: seminorm, $\|\text{const}\|_{\mathcal{B}} = 0$).

Intuitively: Lip from Hyp to Euclidean.

Property: $\tau \in \operatorname{Aut}(\mathbb{D})$, $b \in \mathcal{B} \Rightarrow \|b\|_{\mathcal{B}} = \|b \circ \tau\|_{\mathcal{B}}$.

Thm. $f \in (S) \Rightarrow \|\log f'\|_{\mathcal{B}} \le 6$.

Property. $f \in \mathrm{Aut}(\mathbb{D})$, $b \in \mathcal{B} \Rightarrow \|b\|_{\mathcal{B}} = \|b \circ f\|_{\mathcal{B}}$.

Thm. $f \in (S) \Rightarrow \|\log f'\|_{\mathcal{B}} \le 6$.

Pf. By Bieberbach:
$$r\left|\frac{f''(re^{i\theta})}{f'(re^{i\theta})}\right| \le \frac{2r^2}{1-r^2} + \frac{4r}{1-r^2} \le \frac{6r}{1-r^2}$$

Remark. Univalence criterium (Nehari): $f \in A(\mathbb{D})$, $f' \ne 0$, $\|\log f'\|_{\mathcal{B}} \le 1 \Rightarrow f$ is univalent.

Cor. $\exists C$: $\left|\frac{f'(z_1)}{f'(z_2)}\right| \le C\,\rho(z_1, z_2)$ for $f \in A(\mathbb{D})$, f - conf, $f(0) = 0$.

(S) and Hardy spaces. Def. Hardy space $H^p(\mathbb{D}) = \{f \in A(\mathbb{D}) : \sup \int_0^{2\pi} |f(re^{i\theta})|\,d\theta < \infty\}$.

Pravitz Thm $(S) \subset H^p \quad \forall\ 0 < p < 1/2$

Follows from

Pravitz lemma For $f \in (S)$, $\frac{1}{2\pi}\int |f(r\xi)|^p\,d|\xi| \le p\int_0^r \frac{M(\rho)^p}{\rho}\,d\rho$, where
$M(\rho) = \max_{|z|=\rho} |f(z)|$.

And $M(\rho) \le \frac{\rho}{(1-\rho)^2}$ (Growth Thm).

Pf. (of Pravitz Lemma).
$$\frac{d}{dr}\int_0^{2\pi} |f(re^{i\theta})|^p\,d\theta = \int \frac{\partial |f(re^{i\theta})|^p}{\partial r}\,d\theta = \int p\,|f(re^{i\theta})|^p \frac{\partial \log|f(re^{i\theta})|}{\partial r}\,d\theta$$
$$\underset{\text{Cauchy Riemann for } \log f(re^{i\theta})}{=} \int p\,|f(re^{i\theta})|^p \frac{\partial \arg f(re^{i\theta})}{r\,\partial\theta}\,d\theta \underset{w=f(z)}{=} \frac{1}{r}\int_{f(rS^1)} p\,|w|^p\,d\arg w$$

But $f(rS^1) \subset M(r)\mathbb{D}$, so, by Green Formula,
$\int_{f(rS^1)} |w|^p\,d\arg w \le (M(r))^p \cdot 2\pi$. Integrate now $\frac{1}{2\pi}\int |f(re^{i\theta})|^p\,d\theta = \frac{1}{2\pi}\int_0^r \left(\frac{\frac{\partial}{\partial\rho}\int |f(\rho e^{i\theta})|^p\,d\theta}{\rho}\right)d\rho$